# Chapter 3
# Truncation Error Analysis
# Taylor Tables

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# Taylor Tables of Differencing Schemes

- Notation: Consider $u(x, t)$ for fixed $t$ and $x = j\Delta x$ so that,

  $u(x + k\Delta x) = u(j\Delta x + k\Delta x) = u_{j+k}$.

- The generalized form of the Taylor Series Expansions is given by

$$u_{j+k} = u_j + (k\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(k\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots \quad + \quad \frac{1}{n!}(k\Delta x)^n\left(\frac{\partial^n u}{\partial x^n}\right)_j$$
$$+ \quad \ldots$$

- For example, consider the Taylor series expansion for $u_{j+1}$:

$$u_{j+1} = u_j + (\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots \quad + \quad \frac{1}{n!}(\Delta x)^n\left(\frac{\partial^n u}{\partial x^n}\right)_j$$
$$+ \quad \ldots$$

- Or for $u_{j-2}$:

$$u_{j-2} = u_j + (-2\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(-2\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots \quad + \quad \frac{1}{n!}(-2\Delta x)^n\left(\frac{\partial^n u}{\partial x^n}\right)_j$$
$$+ \quad \ldots$$

## Finite Difference Formulas

- Take the expansion for $u_{j-1}$

$$u_{j-1} = u_j - \Delta x \left( \frac{\partial u}{\partial x} \right)_j + \frac{1}{2} (\Delta x)^2 \left( \frac{\partial^2 u}{\partial x^2} \right)_j + \dots$$

- Rearrange terms to

$$\left( \frac{\partial u}{\partial x} \right)_j - \frac{(u_j - u_{j-1})}{\Delta x} = er_t$$

- Truncation error term is $er_t = +\frac{1}{2} \Delta x \left( \frac{\partial^2 u}{\partial x^2} \right)_j$

- The truncation error $er_t$ is made up of 4 important pieces

$$er_t = \text{Sign} \quad \text{Coefficient} \quad \Delta x^p \quad (p+q)^{th} \text{Derivative}$$

# Taylor Table For the $1^{st}$ Order Backward Difference

- Given

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{(u_j - u_{j-1})}{\Delta x} = er_t$$

- Each term expanded in Taylor Series and placed in a table simplifing algebra.
- Note the multiplication by $\Delta x$ to again simplify the table.

|  | $u_j$ | $\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$ | $\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | $\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$ | $\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$ |
|---|---|---|---|---|---|
| $\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$ |  | 1 |  |  |  |
| $u_{j-1}$ | 1 | $(-1)\frac{1}{1!}$ | $(-1)^2\frac{1}{2!}$ | $(-1)^3\frac{1}{3!}$ | $(-1)^4\frac{1}{4!}$ |
| $-u_j$ | $-1$ | 0 | 0 | 0 | 0 |
| = |  |  |  |  |  |
| $\Delta x \cdot er_t$ | 0 | 0 | $\frac{1}{2}$ | ? | ? |

- Truncation error term $er_t = \frac{1}{2}\Delta x \left(\frac{\partial^2 u}{\partial x^2}\right)_j$ defined from the first non-zero column.

- Don't forget the division by the $\Delta x$ to undo the previous multiplication.

- Order of accuracy is defined as the exponent on the $\Delta x$ term in $er_t$.

# Taylor Table For the $2^{nd}$ Order Central Difference

- Given

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{(u_{j+1} - u_{j-1})}{2\Delta x} = er_t$$

- The Taylor Table

|  | $u_j$ | $\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$ | $\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | $\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$ | $\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$ |
|---|---|---|---|---|---|
| $2\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$ |  | $2$ |  |  |  |
| $u_{j-1}$ | $1$ | $(-1)\frac{1}{1!}$ | $(-1)^2 \frac{1}{2!}$ | $(-1)^3 \frac{1}{3!}$ | $(-1)^4 \frac{1}{4!}$ |
| $-u_{j+1}$ | $-1$ | $-(1)\frac{1}{1!}$ | $-(1)^2 \frac{1}{2!}$ | $-(1)^3 \frac{1}{3!}$ | $-(1)^4 \frac{1}{4!}$ |
| $=$ |  |  |  |  |  |
| $2\Delta x \cdot er_t$ | $0$ | $0$ | $0$ | $-\frac{1}{3}$ | ? |

- The truncation error term $er_t = -\frac{1}{6}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_j$ is defined from the first non-zero column.

- Accuracy is $2^{nd}$ Order.

# Taylor Table For A General 3 Point Difference Scheme

- Starting with

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{1}{\Delta x}(c\,u_{j-2} + b\,u_{j-1} + a\,u_j) = er_t$$

- The Taylor Table

|  | $u_j$ | $\Delta x \cdot$ $\left(\frac{\partial u}{\partial x}\right)_j$ | $\Delta x^2 \cdot$ $\left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | $\Delta x^3 \cdot$ $\left(\frac{\partial^3 u}{\partial x^3}\right)_j$ | $\Delta x^4 \cdot$ $\left(\frac{\partial^4 u}{\partial x^4}\right)_j$ |
|---|---|---|---|---|---|
| $\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$ |  | $1$ |  |  |  |
| $-cu_{j-2}$ | $-c$ | $-c \cdot (\text{-}2) \cdot \frac{1}{1}$ | $-c \cdot (\text{-}2)^2 \cdot \frac{1}{2}$ | $-c \cdot (\text{-}2)^3 \cdot \frac{1}{6}$ | $-c \cdot (\text{-}2)^4 \cdot \frac{1}{24}$ |
| $-bu_{j-1}$ | $-b$ | $-b \cdot (\text{-}1) \cdot \frac{1}{1}$ | $-b \cdot (\text{-}1)^2 \cdot \frac{1}{2}$ | $-b \cdot (\text{-}1)^3 \cdot \frac{1}{6}$ | $-b \cdot (\text{-}1)^4 \cdot \frac{1}{24}$ |
| $-au_j$ | $-a$ |  |  |  |  |
| $=$ |  |  |  |  |  |
| $\Delta x \cdot er_t$ | ? | ? | ? | ? | ? |

- Now instead of having colums sum to zero, we set enough colums to zero to satisfy the number of unknowns.

# Taylor Table For A General 3 Point Difference Scheme

- This time the first three columns sum to zero if

$$\begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 0 \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

- Note we put the linear equations into a matrix form, let Matlab do the work for you.

- Which gives $[c, b, a] = \frac{1}{2}[1, -4, 3]$.

- In this case the fourth column provides the leading truncation

$$er_t = \frac{1}{\Delta x}\left[\frac{8c}{6} + \frac{b}{6}\right]\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_j = \frac{\Delta x^2}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j$$

- Thus we have derived a second-order backward-difference approximation of a first derivative:

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{2\Delta x}(u_{j-2} - 4u_{j-1} + 3u_j) + O(\Delta x^2)$$

# Taylor Table For Other Derivatives, e.g. $2^{nd}$

- Consider a gerneral 3 point formula for the $2^{nd}$ derivative

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j - \frac{1}{\Delta x^2}\left(a\,u_{j-1} + b\,u_j + c\,u_{j+1}\right) = er_t$$

- The Taylor Table is

| | $u_j$ | $\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$ | $\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | $\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$ | $\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$ |
|---|---|---|---|---|---|
| $\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$ | | | $1$ | | |
| $-a\,u_{j-1}$ | $-a$ | $-a \cdot (-1) \cdot \frac{1}{1}$ | $-a \cdot (-1)^2 \cdot \frac{1}{2}$ | $-a \cdot (-1)^3 \cdot \frac{1}{6}$ | $-a \cdot (-1)^4 \cdot \frac{1}{24}$ |
| $-b\,u_j$ | $-b$ | | | | |
| $-c\,u_{j+1}$ | $-c$ | $-c \cdot (1) \cdot \frac{1}{1}$ | $-c \cdot (1)^2 \cdot \frac{1}{2}$ | $-c \cdot (1)^3 \cdot \frac{1}{6}$ | $-c \cdot (1)^4 \cdot \frac{1}{24}$ |
| $=$ | | | | | |
| $\Delta x^2 \cdot er_t$ | ? | ? | ? | ? | ? |

- Setting the first 3 colums to 0 leads to

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

- The solution is given by $[a, b, c] = [1, -2, 1]$.

## Taylor Table For $2^{nd}$ Derivative

- In this case $er_t$ occurs at the fifth column in the table (for this example all even columns will vanish by symmetry) and one finds

$$er_t = \frac{1}{\Delta x^2} \left[ \frac{-a}{24} + \frac{-c}{24} \right] \Delta x^4 \left( \frac{\partial^4 u}{\partial x^4} \right)_j = \frac{-\Delta x^2}{12} \left( \frac{\partial^4 u}{\partial x^4} \right)_j$$

- Note that $\Delta x^2$ has been divided through to make the error term consistent.

- We have just derived the familiar 3-point central-differencing point operator for a second derivative

$$\left( \frac{\partial^2 u}{\partial x^2} \right)_j = \frac{1}{\Delta x^2} (u_{j-1} - 2u_j + u_{j+1}) + O(\Delta x^2)$$